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Conformal SO(2,4) transformations for the helical AdS string solution

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Abstract: By applying the conformal SO(2,4) transformations to the folded rotating string configuration with two spins given by a certain limit from the helical string solution in $AdS_3 \times S^1$, we construct new string solutions whose energy-spin relations are characterized by the boost parameter. When two SO(2,4) transformations are performed with two boost parameters suitably chosen, the straight folded rotating string solution with one spin in AdS_3 is transformed in the long string limit into the long spiky string solution whose expression is given from the helical string solution in AdS_3 by making a limit that the modulus parameter becomes unity.

Keywords: AdS-CFT Correspondence, Bosonic Strings.

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Contents

1. Introduction

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the string theory in $AdS_5 \times S^5$, where various types of classical string solutions play an important role. The energy spectrum of certain string states matches with the spectrum of dimensions of field theory operators in the SYM theory [2, 3]. There has been a mounting evidence that the spectrum of AdS/CFT is described by studying the multi-spin folded or circular rotating string solutions in $AdS_5 \times S^5$ in a particular large spin limit [4−6] and by analyzing the Bethe equation for the diagonalization of the integrable spin chain in the planar SYM theory [7−9].

The other class of string solutions in AdS_5 have been constructed to have a number of spikes on the string and be associated with higher twist operators in SYM [10]. The spiky string solution has been generalized to a string configuration in sphere [11]. By the T-duality transformation $\tau \leftrightarrow \sigma$ of the spiky string solutions in AdS_5 and $R \times S^5$ new dual spiky string solutions have been produced [12, 13].

In another large spin limit such that both the spin chain and the string effectively become very large there has been a construction of a rotating open string solution with one spin in $R \times S^2$, namely, the giant magnon [14] which is a particular case of the spiky string in $R \times S^2$ [11]. The one-spin giant magnon has been identified with an elementary magnon excitation in the long spin chain where the energy-spin relation agrees with the strong 't Hooft coupling limit of that for the spin chain magnon derived from the SU(2|2) × SU(2|2) supersymmetry with a novel central extension [15]. The scattering of two giant magnons has been analyzed such that the giant magnon is identified with the sine-Gordon soliton. By exploiting the equivalence between the string theory in $R \times S^3$ and the complex sine-Gordon theory via Pohlmeyer's reduction [16] the dispersion relation for the two-charge dyonic giant magnon has been constructed [17]. There have been various investigations such as the multi-spin giant magnons [18−22], the S-matrix of giant magnons [23], the finite size corrections to the giant magnon dispersion relations [18, 24] and the semiclassical quantization of giant magnons [25].

Through the equivalence between the O(4) string sigma model and the complex sine-Gordon model, a more general string solution in $R \times S^3$, namely, the helical string has been constructed [26] in terms of the elliptic theta functions and two parameters, the soliton velocity v and the elliptic modulus parameter k, to interpolate between the two-spin folded/circular string in the $v \to 0$ limit and the dyonic giant magnon in the $k \to 1$ limit [17]. There are two kinds of helical strings, the type (i) helical string characterized by the number of spikes and the type (ii) one by the number of crossing the equator of sphere. The helical string solution has been reconstructed [27] as a finite-gap solution [28] in the algebro-geometric approach to the string equations of motion [29].

By perfoming the T-duality transformation $\tau \leftrightarrow \sigma$ to this helical string in $R \times S^3$, a new general string solution has been presented [30] to interpolate between the pulsating string and the dual spiky string with single spike [12]. From the type (i) helical string in $R \times S^3$, the type (iii) helical string in $AdS_3 \times S^1$ with two spins has been produced by the analytic continuation of the elliptic modulus squared, while the type (iv) helical string has been generated by shifting the boosted worldsheet space coordinate. The type (iii) helical string becomes the folded rotating string as well as the spiky string [10] in certain parameter limits, and the type (iv) helical string includes the SL(2) giant magnon [19, 20] in the $k \to 1$ limit. The finite-size correction to the dyonic giant magnon has been computed by analyzing the asymptotic behavior of the type (i) helical string with two spins in $R \times S^3$ [31].

On the other hand in order to study the planar 4-gluon amplitude at strong coupling in the SYM theory the open string solution in AdS_5 with Euclidean worldsheet has been constructed [32] for computing the Wilson loop with 4 cusps in the T-dual coordinates whose boundary conditions are determined by the massless gluon momenta. The 4-cusp Wilson loop surface is related by a certain conformal SO(2,4) transformation to the 1-cusp Wilson loop surface found in [33]. There have been various investigations of the Euclidean open string solutions in AdS_5 associated with the planar gluon amplitudes [34 – 36].

In ref. [37] starting from the long string limit of the folded rotating closed string solution with Minkowski worldsheet in AdS_5 [3], the analytic continuation has been taken to yield an open string solution with Euclidean worldsheet, which is further transformed by a discrete SO(2,4) rotation into the 4-cusp Wilson loop solution [32]. The 1-loop correction [37] and the 2-loop correction without spin J in S^5 [38] and with spin J [39] to the classical string solution associated with the 1-cusp Wilson loop surface have been computed, where the cusp anomaly function is derived in the strong-coupling expansion. By taking the infinite spin limit of the spiky rotating string in AdS_3, the spiky solution has been described in a simple analytic expression [40], where a generic arc of the spiky string connecting two spikes reaching the AdS boundary is related by the SO(2,4) boosts to the infinite spin limit [4, 41] of the straight folded rotating string [3]. The SO(2) rotated 1-cusp Wilson loop solution in AdS_5 [37, 33] has been generalized to the case in $AdS_5 \times S^5$ [42]. There has been another study to generate classical string solutions in AdS_5 by using the Pohlmeyer reduction, where the 4-cusp Wilson loop solution associated with the long string limit of the straight folded rotating closed string is related to the sinh-Gordon vacuum [43].

We will consider the SO(2,4) transformations of the type (iii) helical string solution in $AdS_3 \times S^1$. The SO(2,4) transformation with an arbitrary boost parameter will be applied to the straight folded rotating string in $AdS_3 \times S^1$ which is given by a particular limit of the type (iii) helical string solution. The generated string configuration will be shown to satisfy the string equations of motion and its energy-spin relation will be derived. We will perform particular SO(2,4) boosts to the long string limit of the spiky string solution in AdS_3 whose expression is produced from the type (iii) helical string. The SO(2,4) boosted string configuration will be shown to have relation with the 4-cusp Wilson loop solution.

2. SO(2,4) transformations of the folded rotating string

We consider a closed string in $AdS_3 \times S^1$ whose embedding coordinates η_0, η_1 and ξ_1 are expressed as

$$\begin{aligned} \eta_0 &= Y_0 + iY_{-1} = \cosh\rho e^{it}, \qquad \eta_1 = Y_1 + iY_2 = \sinh\rho e^{i\phi_1}, \\ \xi_1 &= X_1 + iX_2 = e^{i\varphi_1} \end{aligned} \tag{2.1}$$

and obey

$$\begin{aligned} \vec{\eta}^* \cdot \vec{\eta} &\equiv -|\eta_0|^2 + |\eta_1|^2 = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1, \\ |\xi_1|^2 &= X_1^2 + X_2^2 = 1. \end{aligned} \tag{2.2}$$

The Polyakov action becomes

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \left[\gamma^{ab}(\partial_a\vec{\eta}^* \cdot \partial_b\vec{\eta} + \partial_a\xi_1^*\partial_b\xi_1) + \tilde{\Lambda}(\vec{\eta}^* \cdot \vec{\eta} + 1) + \Lambda(\xi_1^*\xi_1 - 1)\right], \tag{2.3}$$

from which the equations of motion in the conformal gauge are provided by

$$\partial_a\partial^a\vec{\eta} - (\partial_a\vec{\eta}^* \cdot \partial^a\vec{\eta})\vec{\eta} = 0, \qquad \partial_a\partial^a\xi_1 + (\partial_a\xi_1^*\partial^a\xi_1)\xi_1 = 0 \tag{2.4}$$

and the Virasoro constraints are given by

$$\begin{aligned} 0 &= T_{\sigma\sigma} = T_{\tau\tau} = \frac{\delta^{ab}}{2}(\partial_a\vec{\eta}^* \cdot \partial_b\vec{\eta} + \partial_a\xi_1^*\partial_b\xi_1), \\ 0 &= T_{\tau\sigma} = \mathrm{Re}(\partial_\tau\vec{\eta}^* \cdot \partial_\sigma\vec{\eta} + \partial_\tau\xi_1^*\partial_\sigma\xi_1). \end{aligned} \tag{2.5}$$

Let us devote ourselves to the type (iii) helical string solution in $AdS_3 \times S^1$ with two spins (S, J) in ref. [30], which is expressed in terms of a modulus parameter q and Jacobi theta and zeta functions, $\Theta_\mu(z,q), Z_\mu(z,q)$ as

$$\begin{aligned} \eta_0 &= \frac{C}{\sqrt{qq'}}\frac{\Theta_3(0)\Theta_0(\tilde{X} - i\tilde{\omega}_0)}{\Theta_2(i\tilde{\omega}_0)\Theta_3(\tilde{X})}\exp\left(Z_2(i\tilde{\omega}_0)\tilde{X} + i\tilde{u}_0\tilde{T}\right), \\ \eta_1 &= \frac{C}{\sqrt{qq'}}\frac{\Theta_3(0)\Theta_1(\tilde{X} - i\tilde{\omega}_1)}{\Theta_3(i\tilde{\omega}_1)\Theta_3(\tilde{X})}\exp\left(Z_3(i\tilde{\omega}_1)\tilde{X} + i\tilde{u}_1\tilde{T}\right), \\ \xi &= \exp(i\tilde{a}\tilde{T} + i\tilde{b}\tilde{X}), \end{aligned} \tag{2.6}$$

where $q' = \sqrt{1-q^2}$ and the boosted worldsheet coordinates $(\tilde{T}, \tilde{X})$ are defined by

$$\tilde{T} = \frac{\mu(\tau - v\sigma)}{q'\sqrt{1-v^2}} \equiv \frac{\tilde{\tau} - v\tilde{\sigma}}{\sqrt{1-v^2}}, \qquad \tilde{X} = \frac{\mu(\sigma - v\tau)}{q'\sqrt{1-v^2}} \equiv \frac{\tilde{\sigma} - v\tilde{\tau}}{\sqrt{1-v^2}} \tag{2.7}$$

JHEP05(2008)021

with $(\tilde{\tau}, \tilde{\sigma}) = (\mu/q')(\tau, \sigma)$. This type (iii) string solution with the new modulus q is produced by performing a T-transformation that is defined by $k = iq/q'$, to the type (i) helical string solution with the original modulus k. The string equations of motion for $\vec{\eta}$ in (2.4) lead to the following relations

$$\tilde{u}_0^2 = \tilde{U} - (1-q^2)\frac{\mathrm{sn}^2(i\tilde{\omega}_0)}{\mathrm{cn}^2(i\tilde{\omega}_0)}, \qquad \tilde{u}_1^2 = \tilde{U} + \frac{1-q^2}{\mathrm{dn}^2(i\tilde{\omega}_1)}, \tag{2.8}$$

while the Virasoro constraints yield

$$\begin{aligned} \tilde{a}^2 + \tilde{b}^2 &= -q^2 - \tilde{U} - \frac{2(1-q^2)}{\mathrm{cn}^2(i\tilde{\omega}_0)} + 2\tilde{u}_1^2, \\ \tilde{a}\tilde{b} &= iC^2 \left(\frac{\tilde{u}_0}{q^2} \frac{\mathrm{sn}(i\tilde{\omega}_0)\mathrm{dn}(i\tilde{\omega}_0)}{\mathrm{cn}^3(i\tilde{\omega}_0)} + \tilde{u}_1 \frac{\mathrm{sn}(i\tilde{\omega}_1)\mathrm{cn}(i\tilde{\omega}_1)}{\mathrm{dn}^3(i\tilde{\omega}_1)} \right), \end{aligned} \tag{2.9}$$

where the normalization constant C is determined to satisfy $|\eta_0|^2 - |\eta_1|^2 = 1$ as

$$C = \left(\frac{1}{q^2 \mathrm{cn}^2(i\tilde{\omega}_0)} + \frac{\mathrm{sn}^2(i\tilde{\omega}_1)}{\mathrm{dn}^2(i\tilde{\omega}_1)} \right)^{-1/2}. \tag{2.10}$$

The period in the σ direction of the closed string solution is defined from the invariance of theta functions in (2.6) as

$$-l \le \sigma \le l, \qquad \Delta\sigma = \frac{2q' K(q)\sqrt{1-v^2}}{\mu} \equiv 2l \equiv \frac{2\pi}{n} \tag{2.11}$$

and the closedness conditions for the AdS variables are given by

$$\Delta t = 2K(q)(-iZ_2(i\tilde{\omega}_0) - v\tilde{u}_0) + 2n'_{\mathrm{time}}\pi \equiv 0, \tag{2.12}$$

$$\Delta\phi_1 = 2K(q)(-iZ_3(i\tilde{\omega}_1) - v\tilde{u}_1) + (2n'_1 + 1)\pi \equiv \frac{2\pi N_{\phi_1}}{n}, \tag{2.13}$$

where $n = 1, 2, \cdots$ counts the number of periods in $0 \le \sigma \le 2\pi$ and N_{ϕ_1} is the winding number in the ϕ_1 direction and n'_{time}, n'_1 are integers.

The limit $\tilde{\omega}_{0,1} \to 0$ provides $v = 0$ from (2.12) and generates the folded string solution from (2.6), which is expressed as

$$\eta_0 = \frac{1}{\mathrm{dn}(\tilde{\sigma}, q)} e^{i\tilde{u}_0\tilde{\tau}}, \qquad \eta_1 = \frac{q\,\mathrm{sn}(\tilde{\sigma}, q)}{\mathrm{dn}(\tilde{\sigma}, q)} e^{i\tilde{u}_1\tilde{\tau}}, \qquad \xi_1 = \exp(i\sqrt{\tilde{U} - q^2}\tilde{\tau}), \tag{2.14}$$

where $\tilde{u}_0^2 = \tilde{U}$ and $\tilde{u}_1^2 = \tilde{U} + 1 - q^2$.

Now we are ready to consider the conformal SO(2,4) transformation of the folded rotating string solution. The Polyakov action (2.3) and the Virasoro constraints (2.5) are invariant under the isometry group SO(2,4) of AdS_5. To the folded string solution (2.14) we make an SO(2,4) transformation which is an arbitrary boost in the (Y_2, Y_0) plane

$$Y'_2 = \gamma(Y_2 + \hat{v}Y_0), \qquad Y'_0 = \gamma(\hat{v}Y_2 + Y_0) \tag{2.15}$$

with $\gamma = 1/\sqrt{1-\hat{v}^2}$. The transformed string configuration is expressed as

$$\begin{aligned}\eta'_0 &= \cosh\rho' e^{it'} = \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[\gamma(\hat{v}q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau + \cos\tilde{u}_0\tilde\tau) + i\sin\tilde{u}_0\tilde\tau\right],\\ \eta'_1 &= \sinh\rho' e^{i\phi'_1} = \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[q\mathrm{sn}(\tilde\sigma)\cos\tilde{u}_1\tilde\tau + i\gamma(q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau + \hat{v}\cos\tilde{u}_0\tilde\tau)\right]. \end{aligned} \tag{2.16}$$

When we substitute this string configuration into the string equations of motion, $\partial_a\partial^a\vec{\eta}' - (\partial_a\vec{\eta}'^* \cdot \partial^a\vec{\eta}')\vec{\eta}' = 0$, we obtain the following expression in terms of the rescaled worldsheet variables $(\tilde\tau, \tilde\sigma)$

$$-\partial^2_{\tilde\tau}\vec{\eta}' + \partial^2_{\tilde\sigma}\vec{\eta}' = \frac{\tilde{U}\mathrm{dn}^2(\tilde\sigma) + q^2(\mathrm{cn}^2(\tilde\sigma) - \mathrm{sn}^2(\tilde\sigma)) + q^4\mathrm{sn}^4(\tilde\sigma)}{\mathrm{dn}^2(\tilde\sigma)}\vec{\eta}', \tag{2.17}$$

which shows the invariance of $\partial_a\vec{\eta}^* \cdot \partial^a\vec{\eta}$ under the SO(2,4) transformation. The string equations of motion are confirmed to hold for the terms including a $\hat{v}$ coefficient and the other terms separately, by taking account of $\tilde{u}_0^2 = \tilde{U}, \tilde{u}_1^2 = \tilde{U} + 1 - q^2$ and $\mathrm{dn}^2(\tilde\sigma) = 1 - q^2\mathrm{sn}(\tilde\sigma)$.

The AdS radial coordinate ρ' oscillates in both $\tilde\sigma$ and $\tilde\tau$ such that

$$\begin{aligned}\cosh\rho' &= \frac{\gamma}{\mathrm{dn}(\tilde\sigma)}\left[(\hat{v}q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau + \cos\tilde{u}_0\tilde\tau)^2 + (1-\hat{v}^2)\sin^2\tilde{u}_0\tilde\tau\right]^{1/2},\\ &= \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[1+\gamma^2\left(\hat{v}^2q^2\mathrm{sn}^2(\tilde\sigma)\sin^2\tilde{u}_1\tilde\tau + 2\hat{v}q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau\cos\tilde{u}_0\tilde\tau + \hat{v}^2\cos^2\tilde{u}_0\tilde\tau\right)\right]^{1/2},\\ \sinh\rho' &= \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[q^2\mathrm{sn}^2(\tilde\sigma) + \gamma^2\left(\hat{v}q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau + \cos\tilde{u}_0\tilde\tau\right)^2 - \cos^2\tilde{u}_0\tilde\tau\right]^{1/2}, \end{aligned} \tag{2.18}$$

which are combined to be

$$\cosh 2\rho' = \frac{1+q^2\mathrm{sn}^2(\tilde\sigma)}{\mathrm{dn}^2(\tilde\sigma)} + \frac{2}{\mathrm{dn}^2(\tilde\sigma)}\left[\gamma^2\left(\hat{v}q\mathrm{sn}(\tilde\sigma)\sin\tilde{u}_1\tilde\tau + \cos\tilde{u}_0\tilde\tau\right)^2 - \cos^2\tilde{u}_0\tilde\tau\right]. \tag{2.19}$$

The AdS time t' is represented by

$$\tan t' = \frac{\tan\tilde{u}_0\tilde\tau}{\gamma\left(1 + \hat{v}q\mathrm{sn}(\tilde\sigma)\frac{\sin\tilde{u}_1\tilde\tau}{\cos\tilde{u}_0\tilde\tau}\right)}, \tag{2.20}$$

while the angular coordinate ϕ'_1 is expressed as

$$\cot\phi'_1 = \frac{\cot\tilde{u}_1\tilde\tau}{\gamma\left(1 + \frac{\hat{v}}{q\mathrm{sn}(\tilde\sigma)}\frac{\cos\tilde{u}_0\tilde\tau}{\sin\tilde{u}_1\tilde\tau}\right)}. \tag{2.21}$$

From (2.13) with $n'_1 = 0$, the number of periods n is given by 2 for the $N_{\phi_1} = 1$ case and 4 for $N_{\phi_1} = 2, \cdots$, and so on in the straight folded string. The total periodic boundary conditions of t' and ϕ'_1 for the closed string are satisfied owing to $\mathrm{sn}(\tilde\sigma + 2nK(q)) = \mathrm{sn}(\tilde\sigma)$, where the total string interval is specified by $0 \le \tilde\sigma \le 2nK(q)$.

The conserved charges of the SO(2,4) transformed string configuration are defined by

$$
\begin{aligned}
E &= \frac{\sqrt{\lambda}}{\pi}\mathcal{E} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma \mathrm{Im}({\eta'_0}^{*}\partial_\tau \eta'_0), \\
S &= \frac{\sqrt{\lambda}}{\pi}\mathcal{S} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma \mathrm{Im}({\eta'_1}^{*}\partial_\tau \eta'_1), \\
J &= \frac{\sqrt{\lambda}}{\pi}\mathcal{J} = \frac{n\sqrt{\lambda}}{2\pi}\int_{-l}^{l} d\sigma \mathrm{Im}({\xi_1}^{*}\partial_\tau \xi_1).
\end{aligned} \tag{2.22}
$$

By substituting the solution (2.16) into E and S in (2.22) the string energy is evaluated as

$$
\begin{aligned}
\mathcal{E} &= \frac{n}{2}\int_{-K(q)}^{K(q)} d\tilde{\sigma}\frac{\gamma}{\mathrm{dn}^2(\tilde{\sigma})}\left[\tilde{u}_0 + \hat{v}q\mathrm{sn}(\tilde{\sigma})(\tilde{u}_0\sin\tilde{u}_1\tau\cos\tilde{u}_0\tilde{\tau} - \tilde{u}_1\sin\tilde{u}_0\tilde{\tau}\cos\tilde{u}_1\tilde{\tau})\right] \\
&= \gamma\frac{n\tilde{u}_0}{1-q^2}E(q)
\end{aligned} \tag{2.23}
$$

and the AdS spin is similarly given by

$$
\begin{aligned}
\mathcal{S} &= \frac{n}{2}\int_{-K(q)}^{K(q)} d\tilde{\sigma}\frac{\gamma}{\mathrm{dn}^2(\tilde{\sigma})}\left[q^2\tilde{u}_1\mathrm{sn}^2(\tilde{\sigma}) + \hat{v}q\mathrm{sn}(\tilde{\sigma})(\tilde{u}_1\sin\tilde{u}_1\tilde{\tau}\cos\tilde{u}_0\tilde{\tau} - \tilde{u}_0\sin\tilde{u}_0\tilde{\tau}\cos\tilde{u}_1\tilde{\tau})\right] \\
&= \gamma\frac{n\tilde{u}_1}{1-q^2}\left(E(q) - (1-q^2)K(q)\right).
\end{aligned} \tag{2.24}
$$

We see that the $\tilde{\tau}$-dependent terms in $\mathcal{E}$ and $\mathcal{S}$ vanish through the integration of $\mathrm{sn}(\tilde{\sigma})$ over $\tilde{\sigma}$ so that the energy and the AdS spin of the boosted string solution take indeed constant values. The substitution of ξ_1 in (2.14) into J in (2.22) yields the S^1 spin

$$
\mathcal{J} = n\sqrt{\tilde{U} - q^2}K(q). \tag{2.25}
$$

Therefore the energy-spin relation for the boosted string configuration is given by that for the beginning folded string solution [3, 4, 9] with E and S replaced by E/γ and S/γ respectively. For instance taking a particular limit $\tilde{U} \approx q \ll 1$ for (2.23), (2.24) and (2.25) we derive

$$
E^2 = (\gamma J)^2 + \sqrt{\lambda}\gamma n S \tag{2.26}
$$

owing to $E(q) - (1-q^2)K(q) \approx \pi q^2/4$. In the $J = 0$ case which is given by $\tilde{U} = q^2, \tilde{u}_0 = q, \tilde{u}_1 = 1$, the long folded string limit $q \to 1$ yields the following energy-spin relation

$$
E - S = \frac{\gamma n\sqrt{\lambda}}{2\pi}\ln\left(\frac{\pi}{\gamma n\sqrt{\lambda}}S\right), \tag{2.27}
$$

where the boost factor γ appears together with the folding number n.

Here we consider the other SO(2,4) transformation in the (Y_1, Y_{-1}) plane with an arbitrary boost factor γ

$$
Y'_1 = \gamma(Y_1 + \hat{v}Y_{-1}), \qquad Y'_{-1} = \gamma(\hat{v}Y_1 + Y_{-1}), \tag{2.28}
$$

which leads to the following string configuration

$$\begin{aligned}\eta'_0 &= \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[\cos\tilde u_0\tilde\tau + i\gamma(\sin\tilde u_0\tilde\tau + \hat v q\mathrm{sn}(\tilde\sigma)\cos\tilde u_1\tilde\tau)\right],\\ \eta'_1 &= \frac{1}{\mathrm{dn}(\tilde\sigma)}\left[\gamma(q\mathrm{sn}(\tilde\sigma)\cos\tilde u_1\tilde\tau + \hat v\sin\tilde u_0\tilde\tau) + iq\mathrm{sn}(\tilde\sigma)\sin\tilde u_1\tilde\tau\right],\end{aligned} \tag{2.29}$$

which also obeys the string equations of motion (2.17). The AdS radial coordinate ρ' of the boosted string configuration is characterized by

$$\cosh 2\rho' = \frac{1+q^2\mathrm{sn}^2(\tilde\sigma)}{\mathrm{dn}^2(\tilde\sigma)} + \frac{2}{\mathrm{dn}^2(\tilde\sigma)}\left[\gamma^2\left(\sin\tilde u_0\tilde\tau + \hat v q\mathrm{sn}(\tilde\sigma)\cos\tilde u_1\tilde\tau\right)^2 - \sin^2\tilde u_0\tilde\tau\right], \tag{2.30}$$

which shows a similar form to (2.19). The AdS time t' and the angular coordinate ϕ'_1 are expressed as

$$\begin{aligned}\tan t' &= \gamma\tan\tilde u_0\tilde\tau\left(1+\hat v q\mathrm{sn}(\tilde\sigma)\frac{\cos\tilde u_1\tilde\tau}{\sin\tilde u_0\tilde\tau}\right),\\ \cot\phi'_1 &= \gamma\cot\tilde u_1\tilde\tau\left(1+\frac{\hat v}{q\mathrm{sn}(\tilde\sigma)}\frac{\sin\tilde u_0\tilde\tau}{\cos\tilde u_1\tilde\tau}\right),\end{aligned} \tag{2.31}$$

which are compared with(2.20) and (2.21). The energy and the AdS spin of this (Y_1, Y_{-1}) plane boosted string solution are also described by the same expressions as the $\tilde\tau$-independent results in (2.23) and (2.24) so that its energy-spin relation is the same as that for the (Y_2, Y_0) plane boosted string solution.

3. SO(2,4) transformations of the spiky string

Let us consider the $q \to 1$ limit configuration for the type (iii) helical string solution in $AdS_3 \times S^1$ (2.6) which is specified by $v = 0$ from (2.12) and expressed as

$$\eta_0 = C\cosh(\tilde\sigma - i\tilde\omega_0)e^{i\tilde u_0\tilde\tau}, \qquad \eta_1 = C\sinh(\tilde\sigma - i\tilde\omega_1)e^{i\tilde u_1\tilde\tau}, \qquad \xi_1 = e^{i\tilde a\tilde\tau + i\tilde b\tilde\sigma}, \tag{3.1}$$

where

$$\begin{aligned} C &= (\cos^2\tilde\omega_1 - \sin^2\tilde\omega_0)^{-1/2}, \qquad \tilde u_0^2 = \tilde u_1^2 = \tilde U,\\ \tilde a^2 + \tilde b^2 &= -1 + \tilde U, \qquad \tilde a\tilde b = C^2(\tilde u_0\sin\tilde\omega_0\cos\tilde\omega_0 + \tilde u_1\sin\tilde\omega_1\cos\tilde\omega_1).\end{aligned} \tag{3.2}$$

The spikes of this string configuration reach the AdS boundary and the energy and the AdS spin in (2.22) become divergent.

The AdS radial coordinate ρ of this $q \to 1$ limit string solution is characterized by

$$\cosh 2\rho = C^2\left[(\sin^2\tilde\omega_0 + \cos^2\tilde\omega_1)\sinh^2\tilde\sigma + (\cos^2\tilde\omega_0 + \sin^2\tilde\omega_1)\cosh^2\tilde\sigma\right], \tag{3.3}$$

from which the minimum value of ρ is given by

$$\rho_{\min} = \frac{1}{2}\cosh^{-1}\left(\frac{\cos^2\tilde\omega_0 + \sin^2\tilde\omega_1}{\cos^2\tilde\omega_1 - \sin^2\tilde\omega_0}\right). \tag{3.4}$$

JHEP05(2008)021

The AdS time t and the angular coordinate ϕ_1 are represented by

$$\begin{aligned}\tan t &= \frac{\tan\sqrt{\tilde{U}}\tilde{\tau} - \tan\tilde{\omega}_0 \tanh\tilde{\sigma}}{1 + \tan\tilde{\omega}_0 \tanh\tilde{\sigma}\tan\sqrt{\tilde{U}}\tilde{\tau}}, \\ \cot\phi_1 &= \frac{\cot\sqrt{\tilde{U}}\tilde{\tau} + \tan\tilde{\omega}_1 \coth\tilde{\sigma}}{1 - \tan\tilde{\omega}_1 \coth\tilde{\sigma}\cot\sqrt{\tilde{U}}\tilde{\tau}},\end{aligned} \tag{3.5}$$

whose expressions are compared with those in (2.20) and (2.21) for the (Y_2, Y_0) plane boosted string solution.

Now we analyze the $q \to 1$ limit string configuration in AdS_3 with only AdS spin. Since it is specified by $\tilde{U} = 1$, that leads to $\tilde{a} = \tilde{b} = 0, \tilde{\omega}_0 = -\tilde{\omega}_1$ through (3.2), we express the normalization constant as $C = 1/\sqrt{\cos 2\tilde{\omega}_1}$ to have

$$\begin{aligned}\eta_0 &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}}\cosh(\tilde{\sigma} + i\tilde{\omega}_1)e^{i\tilde{\tau}} \\ &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}}\Big[\cos\tilde{\omega}_1\cosh\tilde{\sigma}\cos\tilde{\tau} - \sin\tilde{\omega}_1\sinh\tilde{\sigma}\sin\tilde{\tau} \\ &\qquad + i(\cos\tilde{\omega}_1\cosh\tilde{\sigma}\sin\tilde{\tau} + \sin\tilde{\omega}_1\sinh\tilde{\sigma}\cos\tilde{\tau})\Big], \\ \eta_1 &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}}\sinh(\tilde{\sigma} - i\tilde{\omega}_1)e^{i\tilde{\tau}} \\ &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}}\Big[\cos\tilde{\omega}_1\sinh\tilde{\sigma}\cos\tilde{\tau} + \sin\tilde{\omega}_1\cosh\tilde{\sigma}\sin\tilde{\tau} \\ &\qquad + i(\cos\tilde{\omega}_1\sinh\tilde{\sigma}\sin\tilde{\tau} - \sin\tilde{\omega}_1\cosh\tilde{\sigma}\cos\tilde{\tau})\Big].\end{aligned} \tag{3.6}$$

The energy-spin relation for this $q \to 1$ string configuration was presented [30] by $E - S = (n\sqrt{\lambda}/2\pi)\ln S$ which reproduces the $n = 2$ result for the straight folded rotating string solution [3] and the arbitrary n result for the spiky string solution [10].

The AdS radial coordinate of this long string solution is compactly expressed as

$$\cosh 2\rho = \frac{\cosh 2\tilde{\sigma}}{\cos 2\tilde{\omega}_1}, \tag{3.7}$$

from which the minimum value of ρ is given by

$$\rho_{\min} = \frac{1}{2}\cosh^{-1}\left(\frac{1}{\cos 2\tilde{\omega}_1}\right). \tag{3.8}$$

The parameter $\tilde{\omega}_1$ is determined from (2.13) with $v = 0$ and $q \to 1$. Since there are the relations among the Jacobi zeta functions

$$\begin{aligned}Z_0(u+v,q) - Z_0(u,q) - Z_0(v,q) &= -q^2\mathrm{sn}(u,q)\mathrm{sn}(v,q)\mathrm{sn}(u+v,q), \\ Z_3(u,q) &= Z_0(u + K(q), q),\end{aligned} \tag{3.9}$$

the $Z_3(i\tilde{\omega}_1)$ in (2.13) is expressed as

$$Z_3(i\tilde{\omega}_1) = Z_0(i\tilde{\omega}_1) + Z_0(K) - \frac{\mathrm{sn}(i\tilde{\omega}_1)\mathrm{cn}(i\tilde{\omega}_1)}{\mathrm{dn}(i\tilde{\omega}_1)}. \tag{3.10}$$

Using $Z_0(K) = 0$ and the following asymptotic expressions in terms of $q = e^{-r} \to 1$

$$
\begin{aligned}
K(e^{-r}) &= -\frac{1}{2}\ln r + \frac{3}{2}\ln 2 - \frac{1}{4}r\ln r + o(r\ln^m r), \\
Z_0(i\tilde{\omega}_1, e^{-r}) &= i\tan\tilde{\omega}_1 - ir\frac{\tilde{\omega}_1 + \sin\tilde{\omega}_1\cos\tilde{\omega}_1}{2\cos^2\tilde{\omega}_1} + \frac{2i\tilde{\omega}_1}{\ln r} + O(r^2), \\
\mathrm{sn}(i\tilde{\omega}_1, e^{-r}) &= i\tan\tilde{\omega}_1 + O(r), \qquad \mathrm{cn}(i\tilde{\omega}_1, e^{-r}) = \frac{1}{\cos\tilde{\omega}_1} + O(r), \\
\mathrm{dn}(i\tilde{\omega}_1, e^{-r}) &= \frac{1}{\cos\tilde{\omega}_1} + O(r),
\end{aligned} \tag{3.11}
$$

which were presented in ref. [30], we derive

$$
\lim_{r\to 0} K(q)Z_3(i\tilde{\omega}_1) = -i\tilde{\omega}_1. \tag{3.12}
$$

For the $N_{\phi_1} = 1$ case with $n_1' = 0$ we have $2\tilde{\omega}_1 = \pi - 2\pi/n$ and the real parameter $\tilde{\omega}_1$ is fixed by $\tilde{\omega}_1 = \pi/6$ for $n = 3$. Since the asymtotic expressions of η_1 at $\tilde{\sigma} = \pm K(1)$ are given by

$$
\eta_1(\tilde{\tau}, \tilde{\sigma} = K(1)) = \frac{e^{K(1)}}{2\sqrt{\cos 2\tilde{\omega}_1}} e^{i(\tilde{\tau} - \tilde{\omega}_1)}, \qquad \eta_1(\tilde{\tau}, \tilde{\sigma} = -K(1)) = \frac{e^{K(1)}}{2\sqrt{\cos 2\tilde{\omega}_1}} e^{i(\tilde{\tau} + \tilde{\omega}_1 - \pi)}, \tag{3.13}
$$

the radial coordinate becomes divergent at $\tilde{\sigma} = \pm K(1)$ and the angle difference $\Delta\phi_1 = \phi_1(\tilde{\sigma} = K(1)) - \phi_1(\tilde{\sigma} = -K(1))$ per period is estimated as $\Delta\phi_1 = \pi - 2\tilde{\omega}_1$. Therefore for the $n = 3$ spiky string case with one winding number $\Delta\phi_1$ becomes $2\pi/3$ consistently. The straight folded string specified by $n = 2, \Delta\phi_1 = \pi$ gives $\tilde{\omega}_1 = 0$.

Here we devote ourselves to the string configuration (3.6) for one arc part between two spikes, which is regarded an open string that reaches the AdS boundary. We perform the following particular two SO(2,4) boosts in the (Y_2, Y_0) and (Y_1, Y_{-1}) planes to the $q \to 1$ limit string solution (3.6)

$$
\begin{aligned}
\begin{pmatrix} Y_2' \\ Y_0' \end{pmatrix} &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}} \begin{pmatrix} \cos\tilde{\omega}_1 & \sin\tilde{\omega}_1 \\ \sin\tilde{\omega}_1 & \cos\tilde{\omega}_1 \end{pmatrix} \begin{pmatrix} Y_2 \\ Y_0 \end{pmatrix}, \\
\begin{pmatrix} Y_1' \\ Y_{-1}' \end{pmatrix} &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}} \begin{pmatrix} \cos\tilde{\omega}_1 & -\sin\tilde{\omega}_1 \\ -\sin\tilde{\omega}_1 & \cos\tilde{\omega}_1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_{-1} \end{pmatrix}
\end{aligned} \tag{3.14}
$$

to obtain

$$
\begin{aligned}
Y_0' &= \cosh\tilde{\sigma}\cos\tilde{\tau}, \qquad Y_{-1}' = \cosh\tilde{\sigma}\sin\tilde{\tau}, \\
Y_1' &= \sinh\tilde{\sigma}\cos\tilde{\tau}, \qquad Y_2' = \sinh\tilde{\sigma}\sin\tilde{\tau}, \qquad Y_0'Y_2' = Y_{-1}'Y_1',
\end{aligned} \tag{3.15}
$$

which give

$$
\eta_0' = \cosh\tilde{\sigma} e^{i\tilde{\tau}}, \qquad \eta_1' = \sinh\tilde{\sigma} e^{i\tilde{\tau}}. \tag{3.16}
$$

This string solution is expressed as $\rho' = \tilde{\sigma}, t' = \tilde{\tau}, \phi_1' = \tilde{\tau}$ so that it approximates the large spin limit of the straight folded rotating closed string with two turning points $(n = 2)$ [3].

JHEP05(2008)021

Thus the long spiky string configuration (3.6) whose shape is characterized by the angle difference $\Delta\phi_1 = \pi - 2\tilde{\omega}_1$ is mapped to the long straight folded string configuration with $\Delta\phi_1 = \pi$ by making the double SO(2,4) boosts (3.14) parametrized with $\tilde{\omega}_1$.

We put $\tau \to -i\tau$ for the solution (3.15) with Minkowski worldsheet in order to obtain the string solution with Euclidean worldsheet, and interchange Y_2' and Y_{-1}' through $Y_{-1}' = -iY_2'', Y_2' = -iY_{-1}'', Y_1' = Y_1'', Y_0' = Y_0''$, which is a discrete SO(2,4) transformation, to have

$$\begin{aligned} Y_2'' &= \cosh\tilde{\sigma}\sinh\tilde{\tau}, \qquad Y_{-1}'' = \sinh\tilde{\sigma}\sinh\tilde{\tau}, \\ Y_1'' &= \sinh\tilde{\sigma}\cosh\tilde{\tau}, \qquad Y_0'' = \cosh\tilde{\sigma}\cosh\tilde{\tau}, \qquad Y_0''Y_{-1}'' = Y_2''Y_1''. \end{aligned} \tag{3.17}$$

The resulting open string solution associated with two spikes corresponds to the 4-cusp Wilson loop solution in the string theory computation of the planar 4-gluon scattering amplitude [32]. This 4-cusp Wilson loop solution is transformed by an SO(2,4) rotation back to the elementary 1-cusp solution whose open string world surface ends on two semi infinite lightlike lines steming from one cusp in the T-dual coordinates.

In ref. [40] the SO(2,4) relation between the spiky closed string solution with n spikes and the straight folded rotating solution in AdS_3 in the infinite spin limit has been argued, where the spiky solution in the infinite spin limit is constructed as

$$t = \tau, \qquad \phi_1 = \tau + \sigma, \qquad \cosh 2\rho = \frac{\cos\sigma}{\sqrt{\sin^2\sigma_0 - \sin^2\sigma}} \tag{3.18}$$

with $2\sigma_0 = 2\pi/n$, which show $\rho \to \infty$ at $\sigma \to \pm\sigma_0$ and are compared with (3.5) and (3.7). From (3.18) the minimum value of ρ is given by

$$\rho_{\min} = \frac{1}{2}\cosh^{-1}\left(\frac{1}{\sin\sigma_0}\right), \tag{3.19}$$

which is expressed through $\Delta\phi_1 = \pi - 2\tilde{\omega}_1 = 2\sigma_0$ as

$$\rho_{\min} = \frac{1}{2}\cosh^{-1}\left(\frac{1}{\cos\tilde{\omega}_1}\right), \tag{3.20}$$

which shows a similar value to (3.8) with a slight difference.

Inversely we start with the straight folded closed string in AdS_3 specified by (2.14) with $\tilde{U} = q^2$ and perform a particular SO(2,4) boost (2.15) in the (Y_2, Y_0) plane with $\hat{v} = -\tan\tilde{\omega}_1$ to have (2.16), which has the following $q \to 1$ limit

$$\begin{aligned} \eta_0' &= \frac{1}{\sqrt{\cos 2\tilde{\omega}_1}}(-\sin\tilde{\omega}_1\sinh\tilde{\sigma}\sin\tilde{\tau} + \cos\tilde{\omega}_1\cosh\tilde{\sigma}\cos\tilde{\tau}) + i\cosh\tilde{\sigma}\sin\tilde{\tau} = Y_0' + iY_{-1}', \\ \eta_1' &= \sinh\tilde{\sigma}\cos\tilde{\tau} + \frac{i}{\sqrt{\cos 2\tilde{\omega}_1}}(\cos\tilde{\omega}_1\sinh\tilde{\sigma}\sin\tilde{\tau} - \sin\tilde{\omega}_1\cosh\tilde{\sigma}\cos\tilde{\tau}) = Y_1' + iY_2'. \end{aligned} \tag{3.21}$$

To this string solution we make a special subsequent SO(2,4) transformation corresponding to (2.28), that is, $Y_1'' = \gamma(Y_1' + \hat{v}Y_{-1}'), Y_{-1}'' = \gamma(\hat{v}Y_1' + Y_{-1}')$ with $\hat{v} = \tan\tilde{\omega}_1$ to obtain the string configuration specified by $\eta_0'' = Y_0' + iY_{-1}'', \eta_1'' = Y_1'' + iY_2'$, which turns out to be the long spiky open string configuration expressed by (3.6).

Here we write down the string solution which is derived by performing these double SO(2,4) transformations to the starting folded rotating string solution in AdS_3

$$\begin{aligned}
\eta_0'' &= \frac{\cos\tilde{\omega}_1}{\sqrt{\cos 2\tilde{\omega}_1}} \frac{1}{\mathrm{dn}(\tilde{\sigma})} \left[e^{iq\tilde{\tau}} + i\tan\tilde{\omega}_1 q \mathrm{sn}(\tilde{\sigma}) e^{i\tilde{\tau}} \right], \\
\eta_1'' &= \frac{\cos\tilde{\omega}_1}{\sqrt{\cos 2\tilde{\omega}_1}} \frac{1}{\mathrm{dn}(\tilde{\sigma})} \left[q\mathrm{sn}(\tilde{\sigma}) e^{i\tilde{\tau}} - i\tan\tilde{\omega}_1 e^{iq\tilde{\tau}} \right],
\end{aligned} \tag{3.22}$$

whose $q \to 1$ limit leads to (3.6).

4. Conclusion

To the straight folded rotating string solution in $AdS_3 \times S^1$ extracted from the type (iii) helical closed string solution we have performed the SO(2,4) transformation with an arbitrary boost parameter in the (Y_2, Y_0) or (Y_1, Y_{-1}) plane to construct a new string configuration whose AdS radial coordinate extends with the worldsheet space coordinate and oscillates with the worldsheet time coordinate. The SO(2,4) boosted string configuration has been confirmed to satisfy the string equations of motion and shown to have constant values of the energy and the AdS spin. We have demonstrated that the energy-spin relation is derived from that of the beginning folded rotating string solution simply by scaling the energy and the AdS spin by the boost factor.

We have considered the shape of the long spiky open string solution in AdS_3 which is extracted from the type (iii) helical closed string solution with n spikes in the string sigma model and analyzed how the minimum value of AdS radial coordinate and the angle difference for one arc are characterized by the parameter $\tilde{\omega}_1$ or n. By performing particular double SO(2,4) transformations with the boosts specified by $\tilde{\omega}_1$ in the (Y_2, Y_0) and (Y_1, Y_{-1}) planes to the long spiky solution and making the analytic continuation together with a discrete SO(2,4) transformation, we have produced a long open string solution with Euclidean worldsheet which agrees with the 4-cusp Wilson loop solution in the computation of the 4-gluon planar amplitude. Our expression of the long spiky string solution based on the string sigma model is in a different form from that of ref. [40] where the relation between the spiky solution for the one arc part and the Wilson loop surface solution is presented by using the Nambu-Goto string action. But we have observed that the minimum value of AdS radial coordinate shows a similar expression for the two differently parametrized string solutions.

We have demonstrated that the straight folded rotating string solution in AdS_3 is converted through certain two SO(2,4) rotations specified by the boost velocities $\hat{v} = \pm\tan\tilde{\omega}_1$ with opposite signs into a new string configuration which is expressed by the Jacobi sn, dn functions. In the long string limit it turns out to be the long spiky string solution whose arc shape is characterized by $\tilde{\omega}_1$. In this inverse demonstration we have observed that the specific boost factor γ produces the normalization constant $C = 1/\sqrt{\cos 2\tilde{\omega}_1}$.

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JHEP05(2008)021